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Proximal Markov chain Monte Carlo: Towards building a sparse Earth model

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Problem statement

We want uncertainties on our tomographic models

Probabilistic sampling on global scales is generally too difficult

Current efforts do not allow for certain types of desirable prior information, such as sparsity

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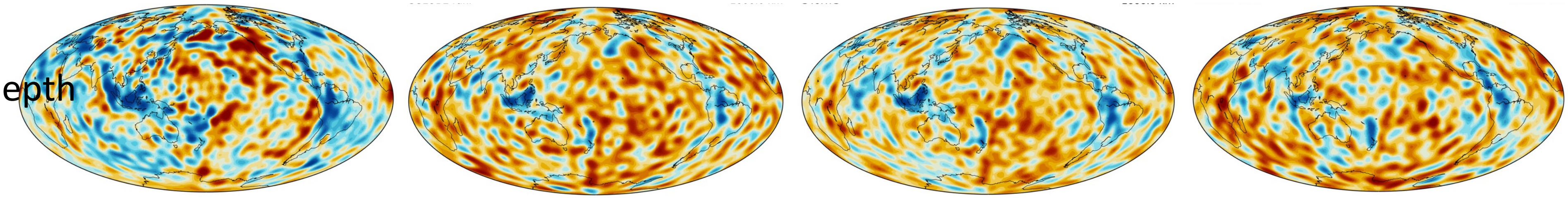
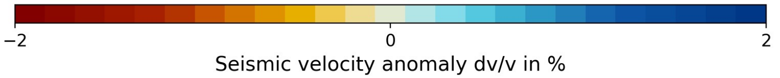


Durand et al., 2017 | Chang et al., 2015 | Ritsema et al., 2011 | French & Romanowicz, 2014



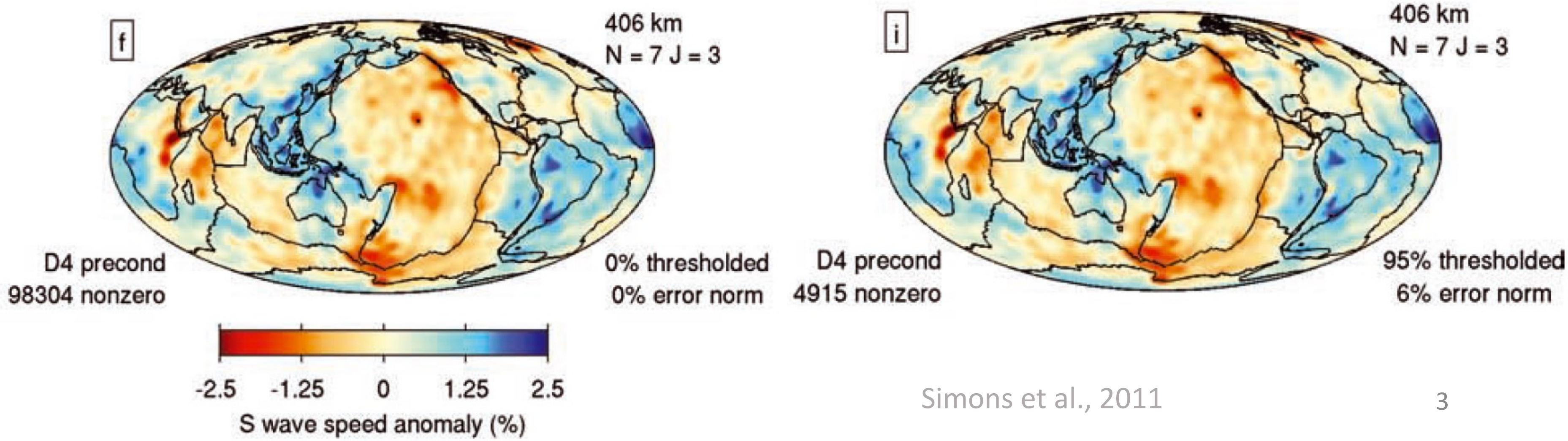
Hosseini et al., 2018

Why sparsity

A signal is “sparse” in a certain basis if many of its expansion coefficients are 0

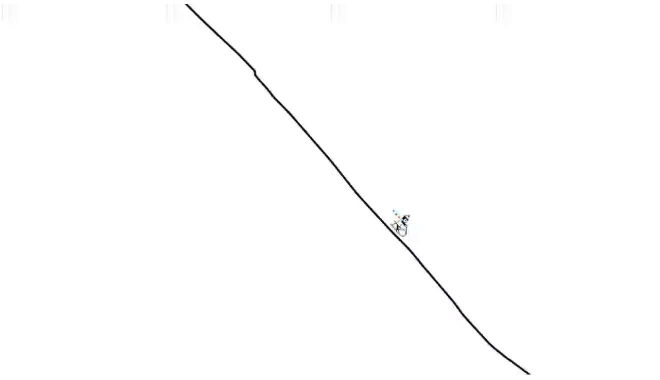
SPARSE IN A WAVELET BASIS! NOT SAYING THE EARTH IS SPARSE

Compressed sensing has shown that signals that are sparse in a certain basis can be accurately recovered from incomplete or poorly distributed data, which is the case in global tomography SHORTEN THIS



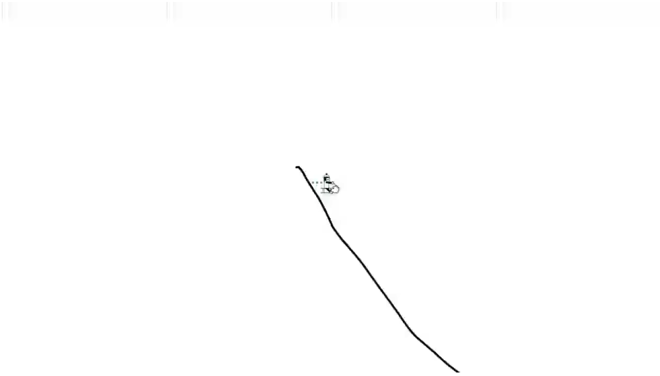
Simons et al., 2011

Proximal algorithms



Convex optimisation techniques using proximity mappings rather than gradients

Particularly well suited to high-dimensional problems like gradient-based approaches



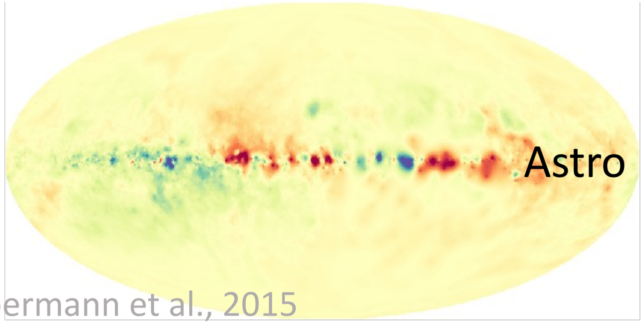
Can be applied to non-smooth problems

What we do

Demonstrate a recent proximal MCMC algorithm on the common problem of building global Rayleigh wave phase velocity maps

We promote sparsity in a spherical wavelet basis

This is the first use of these proximal MCMC methods on spherical problems



Oppermann et al., 2015

VR

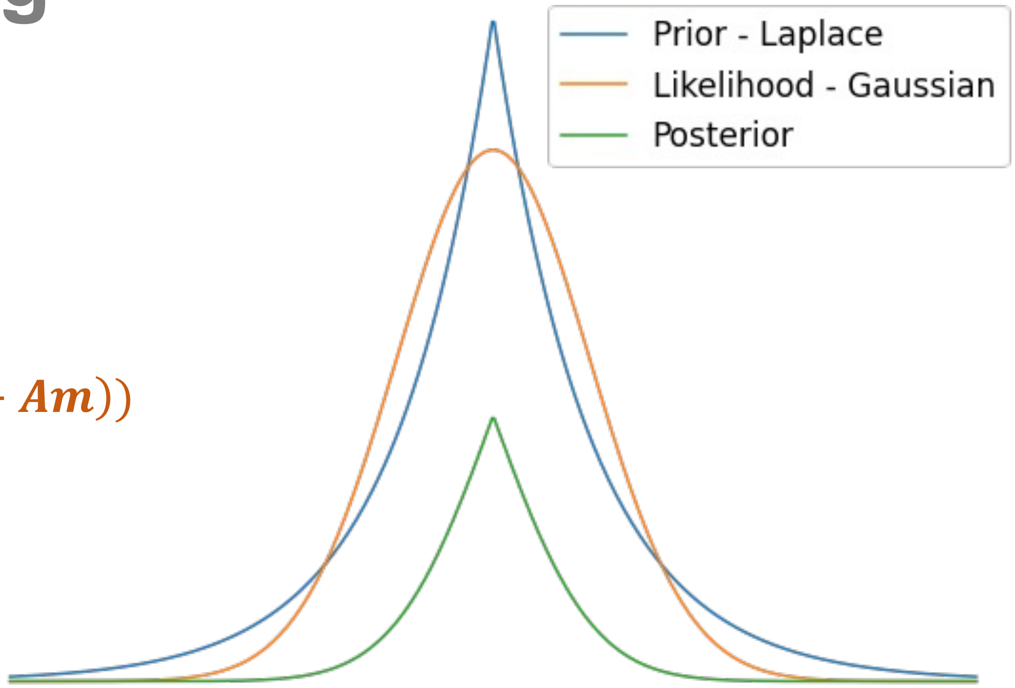
Kageno

Bayesian Sampling

POSTERIOR – what we want
$p(\boldsymbol{m}|\boldsymbol{d}) \propto p(\boldsymbol{d}|\boldsymbol{m})p(\boldsymbol{m})$

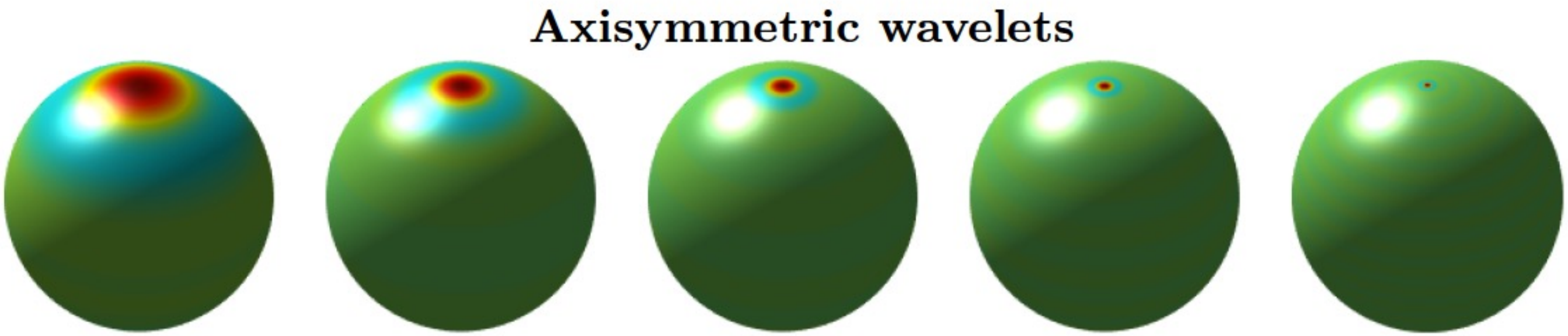
LIKELIHOOD – what we have
$p(\boldsymbol{d}|\boldsymbol{m}) \propto \exp(-(\boldsymbol{d} - \boldsymbol{Am})^T \boldsymbol{C}^{-1}(\boldsymbol{d} - \boldsymbol{Am}))$

PRIOR – what we think
$p(\boldsymbol{m}) \propto \exp(-\mu|\boldsymbol{m}|)$



Wavelet parameterisation

The parameters of our model are the wavelet coefficients in pixel space



Cai et al., 2020, Leistedt et al,. 2013

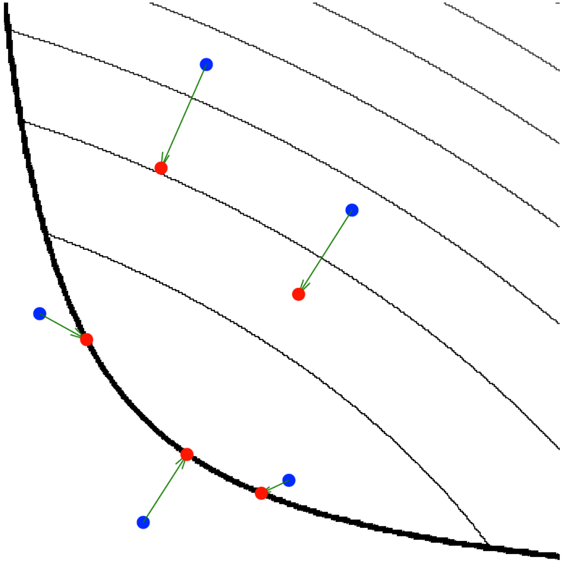
Over 4,000 parameters

Proximity mappings

A gradient step in a smoothed version of a function

This smoothed version is called the λ-MY envelope

Has very useful properties that are similar to the gradient



Parikh & Boyd, 2013

Proximal MCMC

Unadjusted Langevin Algorithm (ULA)

current chain sample

randomness

$$\boldsymbol{m}^{(n+1)} = \boldsymbol{m}^{(n)} + \frac{\delta}{2} \nabla \log \pi\left(\boldsymbol{m}^{(n)}\right) + \sqrt{\delta}\, \boldsymbol{w}^{(n)}$$

next chain sample

gradient of the posterior

colour the terms

Proximal MCMC

Our posterior is of the form $\pi(\boldsymbol{m}) \propto \exp(-g(\boldsymbol{m}) - f(\boldsymbol{m}))$ where f is non-differentiable

Replace f with its smooth λ-MY envelope

current chain sample

$$\boldsymbol{m}^{(n+1)} = \left(1 - \frac{\delta}{\lambda}\right)\boldsymbol{m}^{(n)}$$

next chain sample

randomness

$$-\delta\,\nabla g + \frac{\delta}{\lambda}\operatorname{prox}_f^{\lambda}\left(\boldsymbol{m}^{(n)}\right) + \sqrt{\delta}\,\boldsymbol{w}^{(n)}$$

gradient of the likelihood

prox of the prior

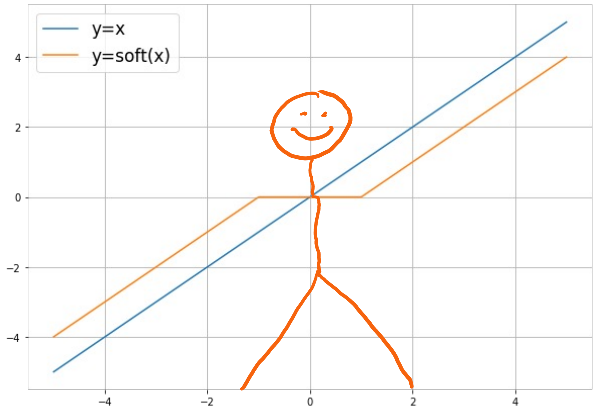
Pereyra, 2016

Moreau-Yosida Unadjusted Langevin Algorithm (MYULA)

Proximal MCMC

So how do we calculate the prox of our prior?

$$f(\boldsymbol{m}) = \mu\|\boldsymbol{m}\|_1 \Longrightarrow \mathrm{prox}_f^{\lambda}(\boldsymbol{m}) = \mathrm{soft}_{\mu\lambda}(\boldsymbol{m})$$

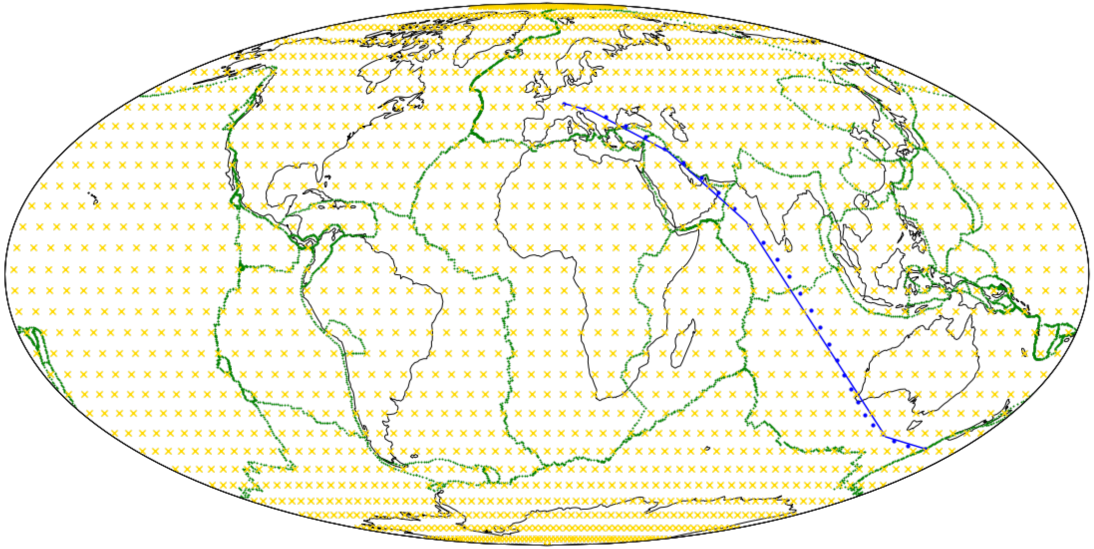


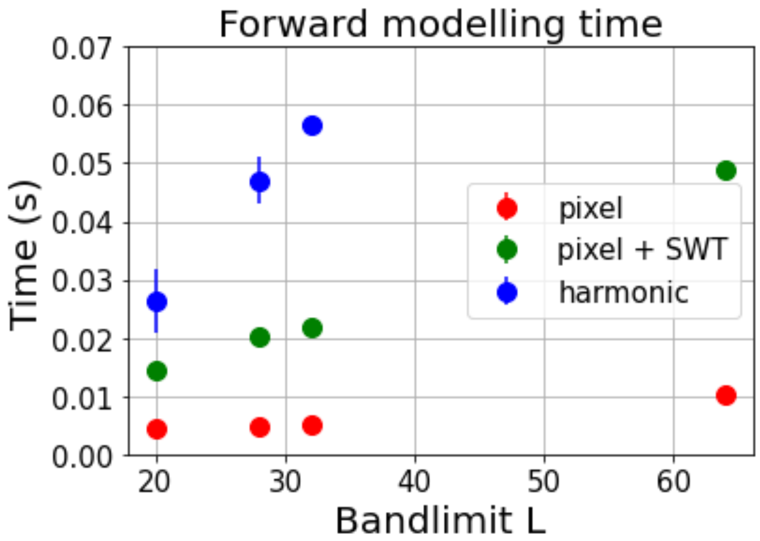
And the gradient of the likelihood?

$$g(\boldsymbol{m}) = \frac{1}{2\sigma^2}\|\boldsymbol{d} - \boldsymbol{A}\boldsymbol{m}\|_2^2 \Longrightarrow \nabla g = \boldsymbol{A}^{\dagger}(\boldsymbol{A}\boldsymbol{m} - \boldsymbol{d})/\sigma^2$$

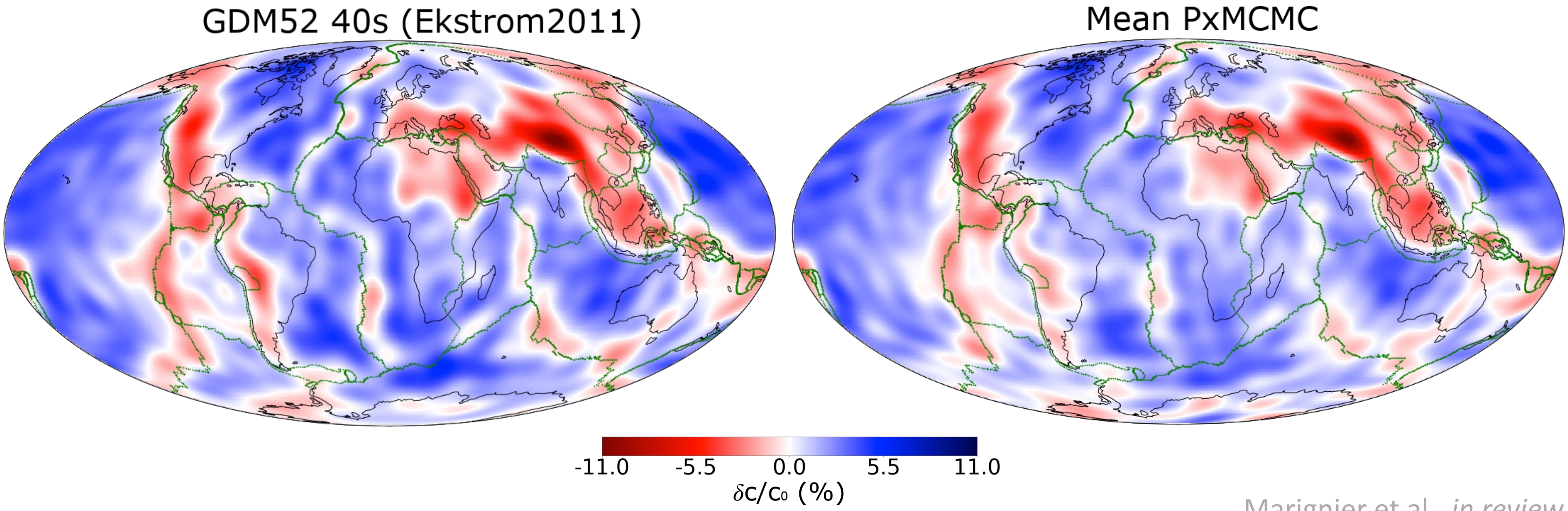
Combettes & Pesquet, 2011

Rayleigh wave phase velocity



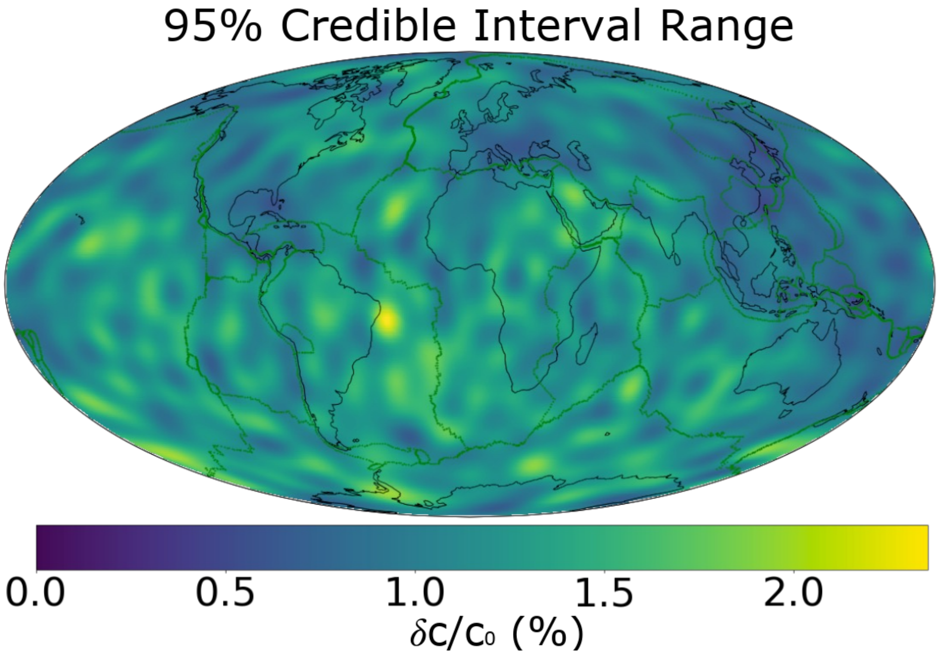


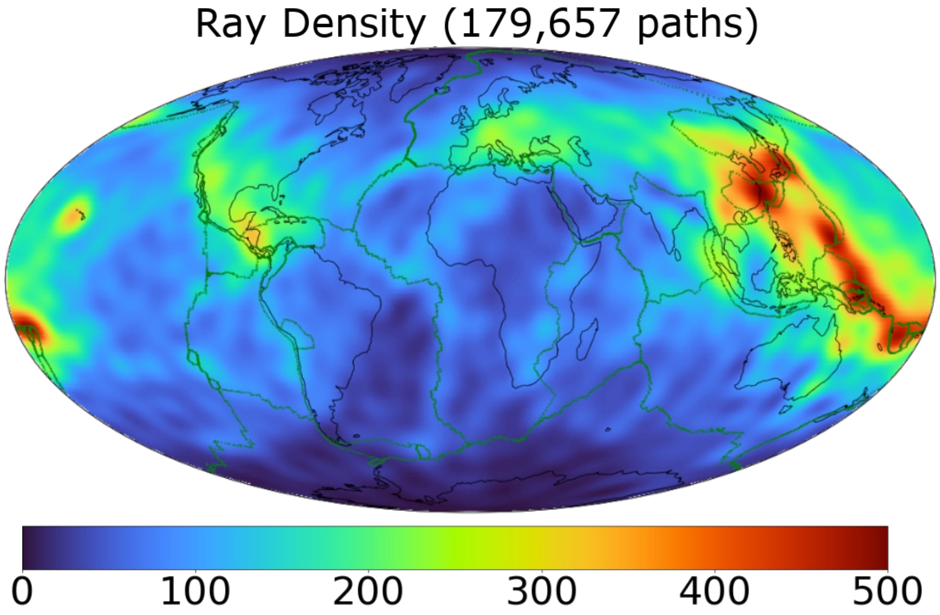
Synthetic Phase velocity experiment



Marignier et al., *in review*

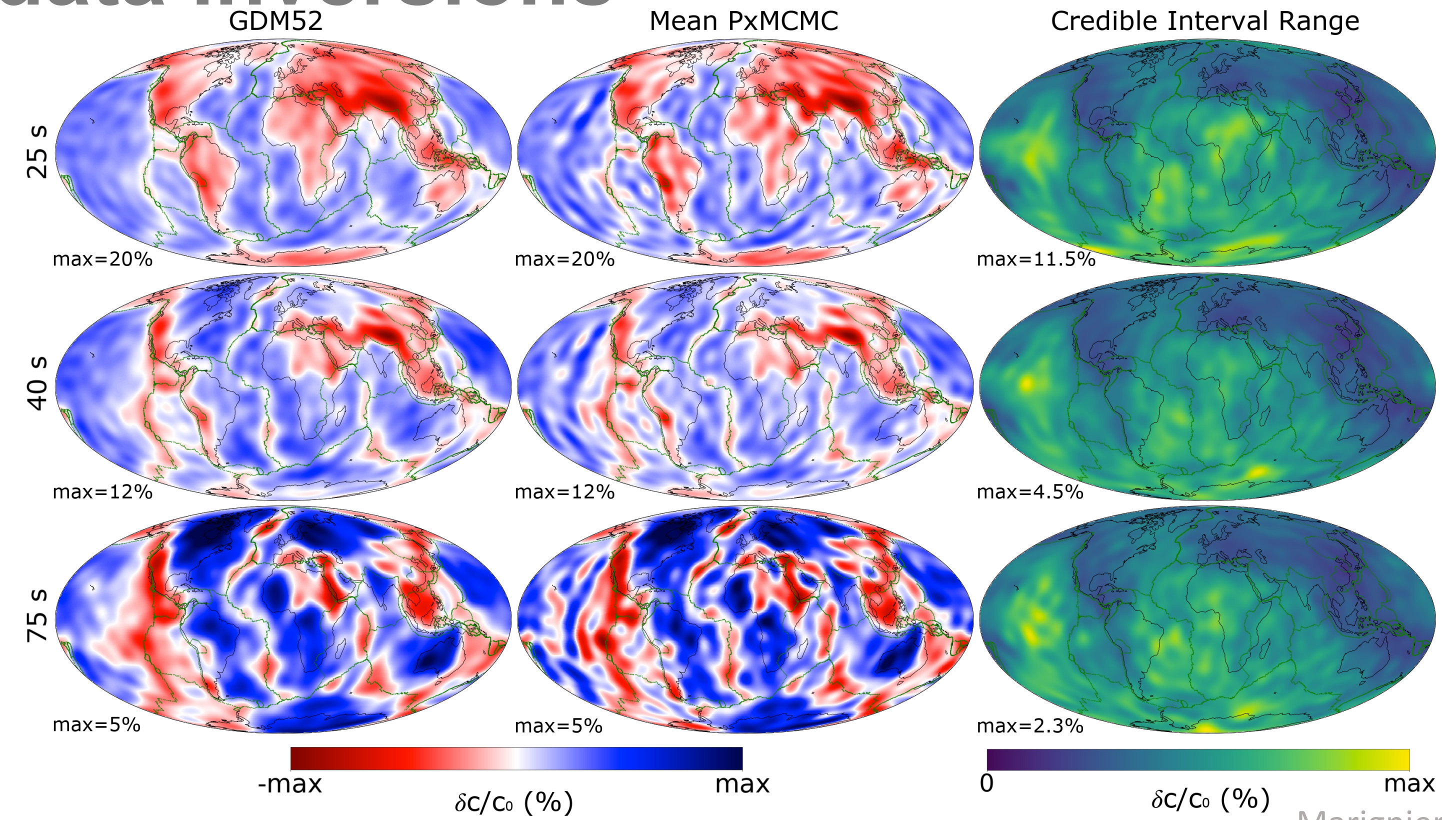
Synthetic Phase velocity experiment





Marignier et al., *in review*

Real data inversions



Marignier et al., *in review*

Future prospects

Current work on proximal methods for 3D inversions, also promoting sparsity in a wavelet basis

This would result in a 3D model with full uncertainty quantification

Hope to get **sharper images** from the compressed sensing approach, upon which we can perform some **hypothesis testing** of features of interest thanks to the uncertainties